

GAUGE THEORY AND THE JONES POLYNOMIAL

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1. INTRODUCTION

One of the most powerful tools for studying isomorphism classes of geometric objects is the theory of invariants. In particular, invariants give a very nice way of telling when two objects are not isomorphic: we find an invariant that takes different values on the two objects. Of course, the effectiveness of this technique relies on our ability to find invariants that can actually distinguish between different objects. In the world of knots and links, the best invariant constructed by the mid-'80s was the HOMFLY polynomial, which give the Conway and Jones polynomials as special cases.

The story of geometric invariants arising in physical contexts is a deep one that touches much of mathematical physics. In classical Maxwell theory, for example, the space of all gauge inequivalent flat vector potentials is precisely the first deRham cohomology, and Chern classes are more sophisticated versions of this. These invariants are connected to classical (non-quantum) physical phenomena; one might ask what kinds of interesting geometric invariants arise from quantum physics. It turns out that knot polynomials kept popping up in a wide array 1+1 dimensional quantum phenomena of interest to theoretical physicists in the late '80s.

The interesting question about both the math and physics of polynomial link invariants is why, despite the fact that links are defined as embeddings of objects into three-dimensional space, the two-dimensional story appears to be much more important. In the mathematics, the link polynomials are defined by giving their values on projections of knots onto the plane via skein relations and then showing that the computation is invariant under the choice of projection and link diagram isotopy. On the physics side, the question is why the link invariants kept appearing in 2-dimensional phenomena. It's completely natural to think of a link as simply the closed paths of several particles in 3-dimensional space; the appearance of link invariants in two-dimensional physics should be more surprising.

This is where superman (aka Ed Witten) comes in. In his 1989 paper *Quantum Field Theory and the Jones Polynomial*, he brought a powerful organizing principle into both the physics and mathematics of link invariants and answered the question alluded to above. In his paper, he studies a topological quantum field theory (TQFT) on 3-manifolds. The connection between two- and three-dimensional pictures arises naturally in the context of TQFTs: we will see that a TQFT assigns a vector space to two-manifolds and linear transformations on vector spaces to bordisms between 2-manifolds. It is this property, together with the behavior of TQFTs under gluing of bordisms, that will give rise to the skein relations for the Jones polynomial.

In this talk, we outline the ideas of Witten's paper. The story will proceed as follows. First, we will do a crash course on links and the HOMFLY polynomial. Next, we'll introduce TQFTs and show how they naturally give rise to invariants of both links and 3-manifolds. This is where the first hint of skein relations will appear. Finally, we'll sketch the construction of Witten's Chern-Simons TQFT; if we have time, we'll be able to see the Jones polynomial explicitly pop out of his framework.

2. CRASH COURSE IN LINKS AND THE JONES POLYNOMIAL

Definition 2.1. A *link* is an embedding of a disjoint union of circles into a 3-manifold M , generally \mathbb{R}^3 or S^3 . \square

Definition 2.2. A *link isotopy* between two links L and L' is a homotopy of homeomorphisms h of the ambient space M of the links such that $h(0, \cdot)$ is the identity, and $h(1, L) = L'$. This is an equivalence relation on links. \square

We are interested only in classifying links up to link isotopy. This is a hard, and central, problem in knot theory.

Definition 2.3. A *link diagram* is the plane projection of a link with crossings (only two-fold crossings are allowed) indicated. \square

A link diagram of a link contains all the information of the link itself, up to link isotopy.

Example 2.1. Link diagrams for the trefoil knot and the Hopf link are shown in figures 2 and 2. \square

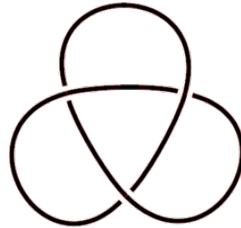


FIGURE 1. The trefoil knot

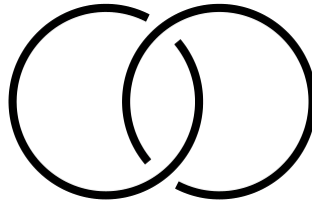


FIGURE 2. The Hopf link

The Jones polynomial and its relatives are computed from link diagrams. These polynomials can be recursively determined from their value on the unknot via a skein relation on the polynomials for Conway triples of links, two concepts which we now define.

Definition 2.4. A triple of oriented links (L_+, L_-, L_0) is a Conway triple if they can be represented by link diagrams D_+, D_-, D_0 which coincide outside a disk in \mathbb{R}^2 and which are respectively isotopic to X_+, X_-, X_0 , as shown in figure 3. \square

The existence and uniqueness of the HOMFLY polynomial, which is a generalization of the Jones polynomial, is guaranteed by the following

Theorem 2.1. *There exists a unique map $L \mapsto P_L$ from the set of all oriented links in \mathbb{R}^3 to the ring $\mathbb{Z}[x, x^{-1}, y, y^{-1}]$ such that*

- *If L and L' are isotopic, then $P_L = P_{L'}$.*

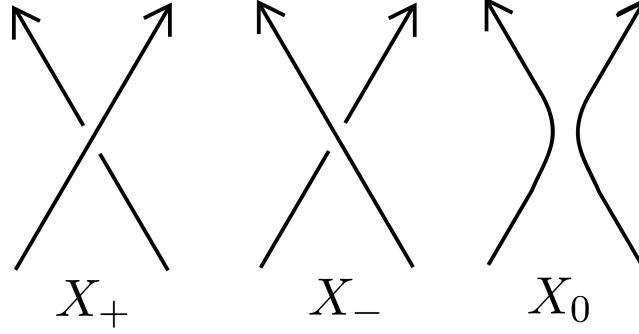


FIGURE 3. Conway triples

- $P_L = 1$ when L is the unknot.
- Whenever (L_+, L_-, L_0) is a Conway triple,

$$(1) \quad xP_{L_+} - x^{-1}P_{L_-} = yP_{L_0} \quad \square$$

The polynomial whose existence is guaranteed by the above theorem is called the *HOMFLY polynomial* and equation 1 is called a *skein relation*. The Jones polynomial and the Conway polynomial are defined as special cases of the HOMFLY polynomial:

Definition 2.5. The *Jones polynomial* $V_L(t)$ and *Conway polynomial* $\nabla_L(z)$ of a link L are defined to be

$$\nabla_L(z) = P_L(1, z), \quad V_L(t) = P_L(t^{-1}, t^{1/2} - t^{-1/2}). \quad \square$$

The Jones polynomial and the Conway polynomial are polynomials satisfying the conditions of the theorem but with skein relations

$$(2) \quad \nabla_{L_+} - \nabla_{L_-} = z\nabla_{L_0}$$

$$(3) \quad t^{-1}V_{L_+} - tV_{L_-} = (t^{1/2} - t^{-1/2})V_{L_0}. \quad \square$$

It is not hard to see that the skein relation determines P_L for an arbitrary link; given a link L , one can obtain a trivial link after changing a finite number of crossings. The skein relation, in turn, determines the value of P_L for trivial links in terms of the unknot polynomial. That this process unambiguously determines the polynomial is a much harder proof; we note only that this part of the proof is much more algebraic, whereas the first half is topological.

Example 2.2. Let O^n denote the n -component trivial link, H the Hopf link, and T the trefoil knot.

$$(4) \quad P_{O^n} = \left(\frac{x - x^{-1}}{y} \right)^{n-1}, \quad P_H = (x^{-1} - x^{-3})y^{-1} + x^{-1}y, \quad P_T = 2x^{-2} - x^{-4} + x^{-2}y^2.$$

The first polynomial is derived by considering the Conway triple (O, O, O^2) and using induction, while the second two polynomials follow from the Conway triples (H, O^2, O) and (T, O, H) . \square

The HOMFLY polynomial can distinguish between knots and their mirror images. In particular, the trefoil knot and its mirror image have different HOMFLY polynomials. In general, if \tilde{L} is the mirror image of L , then $P_{\tilde{L}}(x, y) = P_L(x^{-1}, -y)$.

As a concluding remark, we note that link diagrams aren't a profoundly two-dimensional thing: the crossings capture the relevant 3-dimensional phenomenon. The Chern-Simons TQFT will generate the skein relations for us, but with a slightly more manifest 3-dimensional flavor.

3. TOPOLOGICAL QUANTUM FIELD THEORY

Definition 3.1. Cob_n is the category whose objects are oriented, compact smooth $(n-1)$ -manifolds. A morphism $M \rightarrow N$ is given by a diffeomorphism class of n -manifolds B along with a diffeomorphism $\partial B \simeq \bar{M} \amalg N$, where \bar{M} is the manifold M but with its orientation reversed. B is called a bordism $M \rightarrow N$. Composition of morphisms is given by gluing: if $\partial B \simeq \bar{M} \amalg N$ and $\partial B' \simeq \bar{N} \amalg P$, we can form the composite bordism $B'' = B \amalg_N B'$ by identifying $N \subset \partial B$ with $\bar{N} \subset \partial B'$. The identity bordism $M \rightarrow M$ is given by $M \times [0, 1]$. \square

To be able to give a smooth structure on B'' requires making some choices, like a smooth collar around N , but the diffeomorphism class of the resulting manifold is independent of such choices. The idea is that bordisms represent transitions between spaces or spacetimes. For example, a pair of pants might be understood as two strings scattering into 1 or 1 string splitting into two. Of course, the smooth structure alone gives us no notion of time on the n manifold, so the pair of pants is equally a process of $2 \rightarrow 1$, $1 \rightarrow 2$, $0 \rightarrow 3$ and $3 \rightarrow 0$ string scattering.

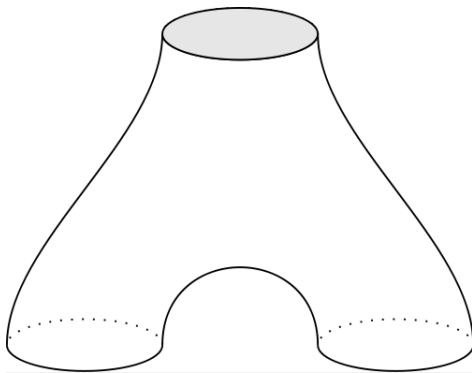


FIGURE 4. A pair of pants is a bordism between three circles.

Before moving on, we remark that Cob_n is a monoidal category: the disjoint union of $(n-1)$ -manifolds gives us a type of tensor product on the category. The empty set is the unit of tensor multiplication.

Definition 3.2. Let \mathcal{H} be the category of complex vector spaces. A topological quantum field theory (TQFT) is a monoidal functor $Z : Cob_n \rightarrow \mathcal{H}$ (monoidal means that $Z(M \amalg N) \simeq Z(M) \otimes Z(N)$). Z is also required to take M to the dual space of $Z(M)$. \square

In other words, Z assigns to each $(n-1)$ -manifold a vector space $Z(M)$ and to each bordism $B : M \rightarrow N$ a linear map $Z(B) : Z(M) \rightarrow Z(N)$. The monoidal property of Z guarantees that $Z(\emptyset) \simeq \mathbb{C}$. Note that if B is a closed n manifold, it can be understood as a bordism $\emptyset \rightarrow \emptyset$, and therefore $Z(B)$ is a linear map $\mathbb{C} \rightarrow \mathbb{C}$, i.e. a number. Because bordisms are defined as diffeomorphism classes of n -manifolds, $B \rightarrow Z(B)$ is an invariant. Moreover, since the identity morphism $M \rightarrow M$ is given by $M \times [0, 1]$ with the natural identification of M with the boundary components, we have $Z(M \times [0, 1]) = Id_{Z(M)}$.

Given any orientation-preserving diffeomorphism $M \simeq N$, we can construct the bordism $M \times [0, 1] : M \rightarrow M$ by identifying M with $M \times \{0\}$ by the identity and M with $M \times \{1\}$ by the diffeomorphism $M \simeq N$. This gives rise to an automorphism $Z(M) \rightarrow Z(M)$, so $Z(M)$ furnishes a natural representation of the group of orientation-preserving diffeomorphisms of M . Actually, if a diffeomorphism ϕ is diffeotopic to the identity, then the bordism constructed from ϕ will be diffeomorphic to the identity bordism, and therefore the same bordism. On the vector space side, this means $Z(M)$ descends to a representation of the group of components of $Diff(M)$.

It is a fact that the vector spaces $Z(M)$ for M are always finite-dimensional. So, we see that the axioms of an n -dimensional TQFT automatically spit out a lot of interesting information about smooth manifolds. For n manifolds, we get an invariant, and for $(n - 1)$ -manifolds M , we get a finite-dimensional representation of the group of components of $Diff(M)$.

We now talk a little bit about the physical interpretation of these axioms. We can think of objects of Cob_n as physical spaces and bordisms as describing time evolution of these spaces (i.e. bordisms \approx spacetimes); $M \times [0, 1]$, for example, can be thought of as the space M propagating without change from time $t = 0$ to $t = 1$. In the absence of this product decomposition, however, there isn't a canonical way to assign a global "time" variable to a bordism, especially since we don't have any additional structure on manifolds that might help (like, e.g., a Lorentzian metric). This is what makes TQFTs "topological" in the physics sense: they depend only on the manifold structure of spacetime and not on any additional structure (e.g. a metric). As for the "quantum" part of TQFTs: this has to do with the fact that $Z(M)$ is a vector space. The space $Z(M)$ can be interpreted as the space of quantum states associated to the space and bordisms give operators which represent the transition amplitude in going from one state to another. As physical theories, TQFTs are in some sense boring because time evolution is trivial: $Z(M \times [0, 1])$ is the identity on $Z(M)$.

So far, knots have made no appearance. To remedy this, we need to slightly modify the axioms of a TQFT. First, specialize to $n = 3$ and choose a compact simple Lie group G . Instead of assigning a number just to a closed 3-manifold B , we now wish to assign a number $Z(B, L, \mu_i)$ to each collection (B, L, μ_i) where L is a link in B , and μ_i is a collection of representations of G , one for each connected component of the link. If B has a boundary, we want to allow only links that intersect transversely with the boundary (maybe we should call these tangles). So, we modify Cob_n to be the category whose objects are 2-manifolds with any number of special signed points marked by some representation of G . We define morphisms $B : M \rightarrow N$ to be the data of a 3-manifold B , oriented link L , a collection of representations μ_i for each connected component of L and an orientation-preserving diffeomorphism $\partial B \simeq \bar{M} \amalg N$ such that the marked points of M and N are precisely the ends of connected components of L and the orientation of the points induced by the orientation on L agrees with signs of those points. Figure 3 gives a picture of the situation we're considering with one dimension reduced.

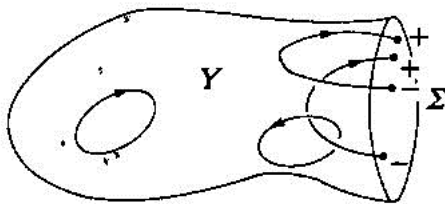


FIGURE 5. The setup for our refined definition of a TQFT.

Physically, we can think of links as a collection of closed paths of a number of particles. The reason that we introduce G and the μ_i is that the particles interact with a gauge field (i.e. a connection on some G -vector bundle over M) and the nature of the interaction is specified by the representation of G under which the particle transforms. Classically, a particle belongs to some representation of group G (in electrodynamics, this is just the charge of the particle), i.e. when a particle is at a point p , its state is described by an element of the fiber over p of some associated G -bundle. This state's time evolution is given by parallel transport with respect to the connection

on the bundle, and the change in the state of a particle after it travels around a closed loop is given by the holonomy of the connection around that loop. The trace of the holonomy is a gauge-invariant quantity, and quantum mechanically, we can ask what the expectation value of the trace of the holonomy is around such loops. This is precisely what our modified QFT should give us.

We are now in a position to understand, at least from the physics side, some of the reasons why the phenomena of two-dimensional physics were so closely linked (pun not intended, at first) to three-dimensional knots. In a TQFT, the information of the n -dimensional physics and the $(n-1)$ -dimensional physics are very closely related. Indeed, with our modified definition of TQFT, we are now considering surfaces with some finite number of marked points. This is precisely the situation that one encounters in conformal field theory or string theory, where one interprets the marked points as certain string states prepared in the infinite past.

Let us now see how skein relations can arise from a TQFT modified as we have just done. Consider figure 3. Given a link L on S^3 , we can isolate any crossing by removing a ball containing only that crossing. So, we've divided S^3 into two manifolds: a ball B containing just two strands L' of the link and its complement. The two manifolds meet at an S^2 with four marked points, two positive and two negative. Z assigns a vector space to this surface, which we'll denote H . If we choose representations μ_i for the components of L (and letting μ_{i_1} and μ_{i_2} denote the representations chosen for the strands of L'), then $Z(B, L', \mu_{i_1}, \mu_{i_2})$ should be an element χ of the vector space H and $Z(S^3 \setminus B, L \setminus L', \mu_i)$ is an element ψ of the dual space H^* . Then, by functoriality of Z , we have

$$(5) \quad Z(S^3, L, \mu_i) = \psi(\chi).$$

Now, if the gauge group under consideration is $SU(N)$ and the representations μ_i are all the fundamental representation, then in the TQFT Witten constructs, the Hilbert space H is two-dimensional. This means, in particular, that there is a linear dependence of any three vectors $\chi, \chi_1, \chi_2 \in H$. In particular, we can let χ_1 be the vector given by a switch in the crossing of L' and χ_2 be the vector given by undoing the crossing entirely. Gluing the three balls back into S^3 gives

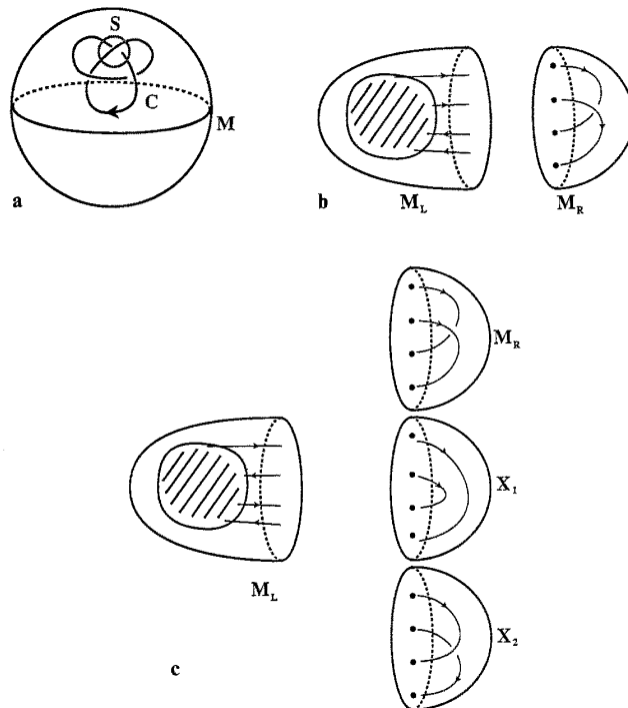


FIGURE 6. A Conway triple from TQFT

a Conway triple (L, L_1, L_2) . Write the linear dependence as

$$\alpha\chi + \beta\chi_1 + \gamma\chi_2 = 0$$

and use the equation 5 to deduce:

$$(6) \quad \alpha Z(S^3, L, \mu_i) + \beta Z(S^3, L_1, \mu_i) + \gamma Z(S^3, L_2, \mu_i) = 0.$$

This has exactly the flavor of a skein relation! The only thing left to be explained is where the parameter t of the Jones polynomial appears. To explain this, we mentioned that Witten actually discovered a family of TQFTs parametrized by an integer k called the level. It is the dependence of the coefficients α, β, γ on k that gives rise to the polynomial nature of the Jones invariant. For general G and μ_i , the TQFT gives other invariants; all will generate skein relations, though the fact that H was two-dimensional made things very easy.

4. CONCLUSION

We have seen how a TQFT can be used to generate a lot of interesting invariants of links and manifolds. The existence of a TQFT like the one we've described is far from obvious, however. We will not endeavor to go through the construction of the Chern-Simons TQFT, since the details on both the physics and math sides are far from obvious. We will, though, sketch the idea of the construction from the physics side.

Let G be a compact simple gauge group, as before. Given a compact, oriented 3-manifold M and a trivial G bundle $E \rightarrow M$, we can choose a connection on E , which can be thought of as a Lie-algebra valued one-form A on M . We construct the Lagrangian, which is a functional of the one-form A :

$$(7) \quad \mathcal{L}_{CS} = \frac{k}{4\pi} \int_M \text{Tr}(A \wedge dA + \frac{2}{3} A \wedge A \wedge A).$$

This Lagrangian is invariant only under gauge transformations homotopic to the identity. If we choose the proper normalization of Tr and integer-valued k , then the action only changes by an integer multiple of 2π . This is enough to define a gauge invariant quantum theory, since the quantization procedure cares only about $\exp(i\mathcal{L})$. In contrast to the classical theory, where k is a parameter that doesn't affect the physics, k affects the quantum physics. Indeed, different values of k give different values of the invariants discussed above. To obtain the Chern-Simons TQFT, we quantize the action in equation 7. The invariants $Z(M, L, \mu_i)$ are given by the vacuum expectation values of products of Wilson line operators in the theory. This is a very rough and very physicsy sketch. The details of the quantization procedure contain a lot of important information about the theory. Unfortunately, we can't go into these details here.

The upshot of this whole story is that physics intuition can lead the way to very interesting new mathematics. Mathematically, knots are simply embeddings of circles in space. To a physicist, knots are trajectories of particles. Particles naturally contain internal properties (like charge, for example) that govern how they interact with a background field, like the electromagnetic field. Properly keeping track of these extra internal properties gives natural generalizations of knot invariants that we already know. This is part of a general story about the inseparability of physics and geometry; both fields are attempts at systematizing intuitions we have about the structure of space and time. This is a relationship that is likely to continue to yield fruitful results.